Levy sections theorem revisited

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Abstract. This paper revisits Levy sections theorem, which generalizes the central limit theorem to encompass autocorrelated variables. Levy sections theorem’s approach is extended to time series and applied to historical daily returns of selected dollar exchange rates. We explain the elevated kurtosis usually observed in such series by their volatilities. In particular, the high kurtosis of emerging markets’ exchange rates is explained by the duration of exchange rate pegs. Thus we suggest an alternative rationale for fat tails that is simpler than those based on either autocorrelations or the presence of Levy distributions.

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1. Introduction

Recently the study of complex systems has attracted the attention of a growing number of physicists. Scaling laws, self-organized criticality, self-similarity, and fractals, just to name a few, have been found in fields as diverse as biology and economics. These phenomena have created the need for a general theoretical framework to explain them coherently, i.e. a physics of complex systems.

One increasing branch of the physics of complex systems is econophysics, which attempts to understand the behavior of financial markets and other economic aspects of ordinary human life. One study of the S&P500 stockmarket index by Mantegna and Stanley [1] is a benchmark for econophysics. Yet the origins of econophysics can be traced back to the 1960s in the research of mathematician Benoit Mandelbrot. Mantegna and Stanley’s work attempted to explain the self-similarity and fat tails observed in financial distributions that can be responsible for a variety of behaviors and, in particular, ultraslow convergence to the Gaussian regime. Here one major contribution was their truncated Levy flights [2], which can explain departures from the central limit theorem and the presence of scaling laws.

More recently, we ourselves have adopted a different line of research [3, 4]. Rather than looking for underlying probability distributions of financial processes, we focused on the role of nonlinear autocorrelations and nonidentically distributed variables. As a result, we could alternatively explain the ultraslow convergence and scaling laws.

This paper moves forward and suggests another alternative yet simpler approach based on Levy sections theorem [5]. The central limit theorem does not take chains of stochastic variables that are autocorrelated into account. Yet Levy sections theorem generalizes the central limit theorem to encompass autocorrelated variables. In a short communication [6] Paul Levy employed his notion of “sections” to outline a proof for a variant of the central limit theorem that considers sums of correlated random variables. Further extensions using less restrictive assumptions were presented later on [7, 8].

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Here we extend the theorem’s approach to time series. To illustrate our framework we borrow from the analytical techniques of time series coming from complex systems. And we take historical daily returns of selected dollar exchange rates from both developed and emerging markets. Such financial series have received increased attention from the physics community in recent years because they properly exemplify a process resulting from nonlinear interaction of many agents. By using the Levy sections to account for local volatilities we find universality in the stochastic behavior of actual financial series. Indeed we explain their stylized fact of elevated kurtosis by the volatilities. Higher kurtosis of emerging markets’ exchange rates ends up explained by the duration of their exchange rate pegs. The longer foreign exchange intervention is, the greater the kurtosis. One can then build a gauge of exchange rate peg duration based on the kurtosis. Thus we present a novel rationale for fat tails that is simpler than those based on either autocorrelation [3, 4] or the presence of Levy distributions.

The rest of the paper is organized as follows. Section 2 presents building-block definitions and Levy sections theorem. Section 3 extends the previous definitions and theorem to time series. Section 4 illustrates our framework using data from exchange rate returns. Section 5 suggests a gauge of foreign exchange intervention using a Gaussian generator. And Section 6 concludes.

2. Definitions and Levy sections theorem

We first consider chains of stochastic variables denoted by \((x_n)_{n \in \mathbb{N}}\). Their conditional probability is \(P\left(x_n \mid x_1, \ldots, x_{n-1}\right)\) and their marginal probability is

\[
p_n(x_n) = \int_{x_1, \ldots, x_{n-1}} P\left(x_n \mid x_1, \ldots, x_{n-1}\right)
\]

where the integral means the sum of all possible walks \(x_1, \ldots, x_{n-1}\). We assume that \(x_n\) is stationary, i.e. \(p_n(x_n) = p(x)\) for all \(n \in \mathbb{N}\). Thus marginal variance of \(x_n\) is

\[
m^2 = \int_{x_n} x_n^2 p(x_n) - \left(\int_{x_n} x_n p(x_n)\right)^2.
\]

Taking sum

\[S_n = x_1 + \ldots + x_n\]

and its respective variance

\[M_n^2 = \langle S_n^2 \rangle - \langle S_n \rangle^2\]

we define “stochastic time” in sequence \((x_n)_{n \in \mathbb{N}}\) as

\[\tau_n = \frac{M_n^2}{m^2}.
\]
The meaning of stochastic time should be straightforward. Take independent \((x_n)_{n \in \mathbb{N}}\), i.e. independent and identically distributed variables. Thus

\[
M_n = mn^{1/2} \Rightarrow \tau_n = n
\]

and stochastic time becomes \(n\). It can be related to “actual time”. Nonindependent sequences mean that linear autocorrelation produces delays (or advances) of stochastic time when compared to actual time. Sometimes \(S_n\) may follow scaling law \(M_n = mn^H\), where \(H\) is Hurst exponent. Here stochastic time is \(\tau_n = n^{2H}\). If \(H > 1/2\) \((H < 1/2)\) stochastic time moves ahead (falls behind) actual time.

Conditional mean of \(x_n\) is

\[
\mu_n = \mathbb{E}_n \left( x_n \right) = \int_{x_n} x_n P \left( x_n \right) dx_n = \int_{x_n} x_n P \left( x_n \right) dx_n
\]

Given that conditional means are stationary and nil, conditional variance of \(x_n\) is

\[
m_n^2 = \mathbb{E}_n \left( x_n^2 \right) = \int_{x_n} x_n^2 P \left( x_n \right) dx_n
\]

where the integral value depends on walk \(x_1, \ldots, x_{n-1}\) in the sequence.

For a given sequence \((x_n)_{n \in \mathbb{N}}\) we can define the following parameters:

\[
\lambda_n = \sum_{i=1}^{n} m_i^2, n \in \mathbb{N}.
\]

For a real, positive \(t\) and an integer \(p\) let

\[
\lambda_p \leq t \leq \lambda_{p+1}
\]

Then we say truncated sequence \((x_1, \ldots, x_p)\) belongs to section \(t\). Section \(t\) is made up of all truncated sequences obeying Eq. (1) to any integer \(p\). These truncated sequences may possess different number of elements. The sum of elements in a truncated sequence belonging to section \(t\) is

\[
S_t = x_1 + \ldots + x_p.
\]

And variance of \(S_t\) is

\[
M_t^2 = \mathbb{E}_t \left( S_t^2 \right) = \left( S_t^2 \right).
\]

We are now ready to introduce Levy sections theorem [6-8].
**Theorem.** For null conditional means $\mu_n = 0$, and bounded stochastic variables $x_n$ (i.e. there exists $U$ such that $|x_n| \leq U, \forall n \in \mathbb{N}$), probability distribution of $S_i / M_i$ is such that

$$
\lim_{j \to \infty} P(S_j < \eta) = \frac{1}{2\pi} \int_{-\infty}^{\eta} e^{-\frac{x^2}{2}} \, dx.
$$

Levy sections theorem generalizes the central limit theorem in that the former considers chains of stochastic variables that are autocorrelated.

Index $n$ in sequence $(x_n)_{n \in \mathbb{N}}$ is related to actual time of a stochastic process, but the sections do not feature a well defined time. Yet this is not so of sum $S_n = x_1 + \ldots + x_n$ in the conventional ordering of sequences in process $(x_n)_{n \in \mathbb{N}}$. Sum $S_j$ converges to a Gaussian but, generally, sum $S_n = x_1 + \ldots + x_n$ does not. However we can define stochastic time $\tau_i$ for $S_j$ the same way we do for $S_n$, i.e.

$$\tau_i = \frac{M_i^2}{m^2}.$$

3. Extending to time series

First consider time series

$$(x_n)_{n=1,\ldots,N}$$

where $n$ is a time counter and $N$ is series size. For positive integer $q$ we can define a new series as

$$(y_n)_{n=1,\ldots,N-2q}$$

where the initial $q$ and last terms of $(x_n)_{n=1,\ldots,N}$ are dropped, i.e.

$$y_n = x_{n+q}, n = 1, \ldots, N-2q.$$

Local mean of size $q$ and time $n$ is

$$\mu_n = \frac{\sum_{i=n}^{n+2q} x_i}{2q + 1}$$

and local volatility is
\[
m_n = \sqrt{\frac{\sum_{i=n}^{n+2q} x_i^2}{2q+1} - \left(\frac{\sum_{i=n}^{n+2q} x_i}{2q+1}\right)^2}.
\]

Here we can employ Levy sections theorem neglecting whether local means are nonstationary and nil. To build up the sections we only take local volatility.

The set of all sums \( S_i \) is

\[
y_i + y_{i+1} + ... + y_{n_i-1} + y_{n_i}, i \in \{1, ..., N - 2q\}
\]

such that

\[
m_i^2 + m_{i+1}^2 + ... + m_{n_i-1}^2 \leq t < m_i^2 + m_{i+1}^2 + ... + m_{n_i}^2.
\]

And the previous quantities defined for section \( t \) can be extended to time series \( (y_n)_{n=1,...,N-2q} \), namely (1) variance \( m^2 = \langle y_n^2 \rangle - \langle y_n \rangle^2 \); (2) ordered \( n \)-sized sum \( S_n \) of the series’ terms; (3) variance of \( S_n \), i.e. \( M_n^2 = \langle S_n^2 \rangle - \langle S_n \rangle^2 \); (4) stochastic time of \( S_n \), i.e. \( \tau_n = M_n^2 / m^2 \); (5) sum \( S_t \) relative to section \( t \); (6) variance of \( S_t \), i.e. \( M_t^2 = \langle S_t^2 \rangle - \langle S_t \rangle^2 \); (7) variance’s stochastic time of \( S_t \), i.e. \( \tau_t = M_t^2 / m^2 \); (8) number of terms of sum \( S_t \), i.e. \( n(S_t) \); and (9) average number of terms of sum \( S_t \), i.e. \( n_t = \langle n(S_t) \rangle \).

4. Illustrating with exchange rate returns

We take historical daily returns of selected dollar exchange rates from six countries, namely Britain, France, Canada, India, Sri Lanka, and China. Source is the Federal Reserve website. Table 1 gives the details about time period and number of observations \( N \).

Then we reckon local volatility of trading weeks (5-day weeks), which means \( q = 2 \) in Section 3’s formulas. Figure 1 shows kurtosis \( K \) as a function of stochastic time. Dashed lines are kurtosis’ evolution of the conventionally ordered series \( S_n \) as a function of \( \tau_n \). Continuous lines represent kurtosis’ evolution of sections \( t \) as a function of \( \tau_t \). To display kurtosis behavior of the sections we first take 100 sections and then let them to vary by steps \( \Delta t \). All sections are produced from the initial \( t = 10^{-15} \) section. The values of \( \Delta t \) used in every currency are in Table 2.

The key features in Figure 1 are as follows. (1) There is kurtosis convergence in the sections of the currencies toward a well defined asymptotic state. This does not hold in the correspondent, conventionally ordered exchange rate time series. (2) Stochastic time of kurtosis convergence is short. Unlike in the conventionally ordered time series, stochastic time of kurtosis convergence is similar for all rates (if we consider \( \tau_t = 10 \) as a threshold for convergence). (3) Kurtosis convergence approaches zero. Developed countries’ currencies present slightly negative kurtosis and emerging countries’ currencies have slightly positive kurtosis. Unlike in the
conventionally ordered series, in the sections the currencies converge to a distribution resembling the Gaussian. (4) Sections’ kurtosis presents a universal behavior for the currencies considered.

A section \( t \)'s sum \( S_t \) is made of the sum of the days with distinct volatilities. While actual time periods do not necessarily exhibit similar volatilities, \( S_t \) for a given \( t \) presents the same volatility. Thus using Levy sections to take into account the effect of local volatilities can uncover universality in the stochastic behavior of actual series.

What happens if we go back to actual time? Assuming stochastic time \( \tau = 10 \) as an equilibrium benchmark, we can take the section \( t \) corresponding to that time for every currency. Table 3 lists values of \( t \) for the exchange rate series. Then we calculate the number of terms of the sums \( S_t \), \( n(S_t) \), and the average number of terms of section \( t \)'s sums, \( n_t \). Figure 2 shows histograms for \( n(S_t) \), and Table 4 presents \( n_t \) for the exchange rates.

Compared to the histograms of emerging markets’ currencies, the histograms of developed markets’ currencies tend to be concentrated at zero. And the average number of days \( n_t \) of the sections \( t \) of developed countries’ currencies is smaller than that of emerging markets’ currencies. These features may be related to the degree of government intervention in the emerging markets’ currencies. A fixed exchange rate regime would mean zero volatility (constant rate) and a return series dominated by zeros. China for instance kept an 11-year-old peg of its currency, the yuan, at 8.28 to the dollar. But there are also four big episodes of revaluation in the yuan-dollar returns’ series considered. This causes an interesting effect. Because volatility nears zero most days, one need to accumulate more days to get a given volatility \( t \) from the Levy section. Table 5 shows the yuan’s \( n_t \) greater than that of the other currencies. Indian and Sri Lankan rupees present smaller values but still greater than those of the pound, French franc, and Canadian dollar. The developed countries’ currencies exhibit very similar \( n_t \). Figure 3 shows histograms related to the currencies’ local volatility. The yuan’s volatility clusters at zero, unlike those of developed countries’ currencies. This explains the observed patterns in the distribution of days (Figure 2).

5. A suggested gauge of exchange rate control

We can also take a Gaussian random generator from reduced variables that are independent and identically distributed (IIDR) [4]. And then we find sequence \( z_n = m_n g_n, n = 1,...,N - 4 \), where \( g_n \) is generated by a normal distribution, and \( m_n \) is local volatility. If \( m_n \) is constant, the distribution of \( z_n = m_n g_n \) collapses to a Gaussian. The column in the middle of Table 5 shows the kurtosis of the IIDR applied to the exchange rates. The right hand side column shows the kurtosis of the original series of daily returns. The effect of local volatilities is unambiguous. Because the generator is Gaussian, elevated kurtosis can be explained by the volatilities.

Thanks to exchange rate pegs, return dispersion is low at the days the rate is fixed. So many return observations fall out of the variance interval. Elevated kurtosis in emerging markets’ exchange rates can then be explained by too many observations outside the variance interval. This rationale for fat tails is simpler than the one based on autocorrelation. The Levy sections filter the effects on local volatility so that the return series present a near-Gaussian universal pattern.

Indeed the exchange rates follow a Gaussian whenever they are floating. Foreign exchange intervention provokes departures from the Gaussian in that it biases volatility evolution. So the greater the control is, the greater the kurtosis. This is so because the pegs tend to bring series dispersion closer to zero, thereby rendering many observations out of the distribution’s variance interval. Thus the kurtosis reckoned in the IIDR can be seen as a gauge of peg duration. Normalizing the pound-dollar’s kurtosis to unity, we can get a relative intervention scale (Table 6).
6. Conclusion

The central limit theorem neglects chains of stochastic variables that are autocorrelated. Levy sections theorem generalizes the latter in that it considers autocorrelated variables too. Though the Levy sections do not feature a well defined time, a stochastic time can be defined for their sum, which converges to a Gaussian.

The Levy sections theorem’s approach can be extended to time series. Here the sections are built up considering local volatility only. We illustrate this with historical daily returns of selected dollar exchange rates, and calculate the local volatilities of their trading weeks. By using the Levy sections to take into account the effect of local volatilities we find universality in the stochastic behavior of actual series.

Unlike in the conventionally ordered exchange rate time series, we find kurtosis convergence toward a well defined asymptotic state in their correspondent Levy sections. We also find the stochastic time of kurtosis convergence to be short. And this is similar for the currencies considered. Moreover the kurtosis convergence approaches zero. In the Levy sections convergence occurs toward a distribution resembling the Gaussian.

Using a Gaussian generator, we explain elevated kurtosis by the volatilities. For instance, an exchange rate peg means no volatility (constant rate) and a return series dominated by zeros. Higher kurtosis of emerging markets’ exchange rates can then be explained by too many observations outside the variance interval, thanks precisely to the duration of exchange rate pegs. This rationale for fat tails is simpler than the one based on autocorrelation.

Indeed the exchange rates follow a Gaussian whenever they are floating. Foreign exchange intervention provokes departures from the Gaussian in that it biases volatility evolution. So the greater the control is, the greater the kurtosis.

We finally suggest a gauge of peg duration based on the kurtosis reckoned in the Gaussian generator.

Acknowledgements

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References


Table 1. Description of data.

<table>
<thead>
<tr>
<th>Country</th>
<th>Currency</th>
<th>Time Period</th>
<th>Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Britain</td>
<td>Pound</td>
<td>4 Jan 71 – 10 Jan 03</td>
<td>8031</td>
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<tr>
<td>France</td>
<td>French Franc</td>
<td>4 Jan 71 – 31 Dec 98</td>
<td>7020</td>
</tr>
<tr>
<td>Canada</td>
<td>Canadian Dollar</td>
<td>4 Jan 71 – 10 Jan 03</td>
<td>8037</td>
</tr>
<tr>
<td>India</td>
<td>Indian Rupee</td>
<td>2 Jan 73 – 10 Jan 03</td>
<td>7525</td>
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<tr>
<td>Sri Lanka</td>
<td>Sri Lankan Rupee</td>
<td>2 Jan 73 – 10 Jan 03</td>
<td>7171</td>
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<tr>
<td>China</td>
<td>Yuan</td>
<td>2 Jan 81 – 10 Jan 03</td>
<td>5471</td>
</tr>
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Table 2. Values for steps $\Delta t$.

<table>
<thead>
<tr>
<th>Currency</th>
<th>$\Delta t$</th>
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<tbody>
<tr>
<td>Pound</td>
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<tr>
<td>Canadian Dollar</td>
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<td>Indian Rupee</td>
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<td>Sri Lankan Rupee</td>
<td>0.02</td>
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<tr>
<td>Yuan</td>
<td>0.00055</td>
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Table 3. Value of $t$ for the section corresponding to $\tau_1 = 10$.

<table>
<thead>
<tr>
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<th>$t$</th>
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<tr>
<td>Indian Rupee</td>
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<tr>
<td>Sri Lankan Rupee</td>
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<tr>
<td>Yuan</td>
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Table 4. Values of $n_t$ for $\tau_1 = 10$.

<table>
<thead>
<tr>
<th>Currency</th>
<th>$n_t$</th>
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</thead>
<tbody>
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<td>Pound</td>
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<td>French Franc</td>
<td>23.13</td>
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<td>Canadian Dollar</td>
<td>17.26</td>
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<td>79.06</td>
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<tr>
<td>Yuan</td>
<td>323.39</td>
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Table 5. Kurtosis of Gaussian IIDR and original series.

<table>
<thead>
<tr>
<th>Currency</th>
<th>IIDR Series’ Kurtosis</th>
<th>Original Series Kurtosis</th>
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<td>6.76</td>
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<tr>
<td>Yuan</td>
<td>1547.7</td>
<td>3486.1</td>
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Table 6. Intervention scale: IIDR series’ kurtosis relative to IIDR pound-dollar return series’ kurtosis.

<table>
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<th>Currency</th>
<th>Intervention Scale</th>
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<td>18.39</td>
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<td>Yuan</td>
<td>228.97</td>
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</table>
Figure 1: Kurtosis (vertical) vs stochastic time. Dashed lines are for conventionally ordered series and continuous lines are for the Levy sections.
Figure 2. Histograms of $n(S_t)$. Section $t$ for each currency corresponds to stochastic time $\tau_i = 10$. 


Figure 3. Distributions of local volatilities of trading weeks. The horizontal axis is normalized by the biggest volatility of a return series.